is transcendental, in particular irrational.

The partial sum

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{p}\right)^{c_k} = \frac{a_n}{b_n}$$

with positive integers a_n and $b_n \le p^{c_n}$ satisfies

$$\begin{aligned} 0 \quad < \quad s - s_n &= \sum_{k=n+1}^{\infty} \left(\frac{1}{p}\right)^{c_k} \le \left(\frac{1}{p}\right)^{c_{n+1}} \sum_{k=0}^{\infty} \left(\frac{1}{p}\right)^{k} \\ &= \frac{1}{p-1}\left(\frac{1}{p}\right)^{c_{n+1}-1} \le \frac{1}{\left(p^{c_n}\right)^{\frac{c_{n+1}-1}{c_n}}}, \end{aligned}$$

because $c_{k+1} - c_k = F_{k+1}F_{k+2} - F_kF_{k+1} = F_{k+1}^2 \ge 1$. Since

$$\lim_{n\to\infty} \frac{c_{n+1}-1}{c_n} = \lim_{n\to\infty} \frac{F_{n+1}F_{n+2}-1}{F_nF_{n+1}} = \lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n} \cdot \frac{F_{n+2}}{F_{n+1}}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2} > 2$$

By the theorem of Thue, Siegel and Roth, for any (fixed) algebraic number x and $\varepsilon > 0$, the inequality

$$0 < \left|x - \frac{a}{b}\right| < \frac{1}{b^{2+\varepsilon}}$$

is satisfied only by a finite number of integers a and b. Hence, s is transcendental.

Also solved by the Kee-Wai Lau, Hong Kong, China (first part of the problem), and the proposer, (first part of the problem)

5432: *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all differentiable functions $f : (0, \infty) \to (0, \infty)$, with $f(1) = \sqrt{2}$, such that

$$f'\left(\frac{1}{x}\right) = \frac{1}{f(x)}, \ \forall x > 0.$$

Solution 1 by Arkady Alt, San Jose, CA

First note that $f'\left(\frac{1}{x}\right) = \frac{1}{f(x)}, \ \forall x > 0 \iff f'(x) = \frac{1}{f\left(\frac{1}{x}\right)}, \ \forall x > 0.$

Then, since $f''(x) = \left(\frac{1}{f\left(\frac{1}{x}\right)}\right)' = -\frac{f'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{f^2\left(\frac{1}{x}\right)}$ and

$\frac{1}{f^2\left(\frac{1}{x}\right)} = \left(f'(x)\right)^2, f'\left(\frac{1}{x}\right) = \frac{1}{f(x)},$

we obtain $f''(x) = \frac{1}{x^2}\left(f'(x)\right)^2 \frac{1}{f(x)} \iff \frac{f(x)f''(x)}{\left(f'(x)\right)^2} = \frac{1}{x^2} \iff$

$\frac{\left(f'(x)\right)^2 - f(x)f''(x)}{\left(f'(x)\right)^2} - 1 = -\frac{1}{x^2} \iff$

$$\left(\frac{f(x)}{f'(x)}\right)' = 1 - \frac{1}{x^2} \iff \frac{f(x)}{f'(x)} = x + \frac{1}{x} + c \iff \frac{f'(x)}{f(x)} = \frac{x}{x^2+cx+1}.$$

Since $f'(1) = \dfrac{1}{f(1)} = \dfrac{1}{\sqrt{2}}$ then $\dfrac{f(1)}{f'(1)} = 1 + \dfrac{1}{1} + c \iff 2 = 2 + c \iff c = 0.$

Therefore, $\dfrac{f(x)}{f'(x)} = x + \dfrac{1}{x} \iff \dfrac{f'(x)}{f(x)} = \dfrac{x}{x^2+1} \iff \ln f(x) = \dfrac{1}{2}\ln\left(x^2+1\right) + d$ and, using $f(1) = \sqrt{2}$

again, we obtain $\ln f(1) = \dfrac{1}{2}\ln\left(1^2+1\right) + d \iff \ln\sqrt{2} = \dfrac{1}{2}\ln 2 + d \iff d = 0.$

Thus, $f(x) = \sqrt{x^2+1}$.

Solution 2 by Albert Stadler, Hirrliberg, Switzerland

The differential equation $f'(x) = \dfrac{1}{f\left(\frac{1}{x}\right)}$ shows that f is differentiable infinitely often in $x > 0$. We differentiate the equation $f'(x)f\left(\dfrac{1}{x}\right) = 1$ and get

$$f''(x)f\left(\frac{1}{x}\right) - f'(x)f'\left(\frac{1}{x}\right)\frac{1}{x^2} = \frac{f''(x)}{f'(x)} - \frac{f'(x)}{f(x)}\frac{1}{x^2} = 0,$$

or equivalently

$$\frac{f''(x)f(x)}{(f'(x))^2} = \frac{1}{x^2}. \qquad (1)$$

By assumption $f(1) = \sqrt{2}$ and thus $f'(1) = \dfrac{1}{f(1)} = \dfrac{\sqrt{2}}{2}$.

We integrate (1) and apply partial integration to get

$$\begin{aligned} 1 - \frac{1}{x} = \int_1^x \frac{dt}{t^2} &= \int_1^x \frac{f''(t)f(t)}{(f'(t))^2}dt \\ &= \int_1^x \frac{d}{dt}\left(\frac{-1}{f'(t)}\right)f(t)dt \\ &= -\left.\frac{f(t)}{f'(t)}\right|_1^x + \int_1^x \frac{f'(t)}{f'(t)}dt \\ &= \frac{f(1)}{f'(1)} - \frac{f(x)}{f'(x)} + x - 1 \\ &= 1 - \frac{f(x)}{f'(x)} + x. \end{aligned}$$

So $\dfrac{f(x)}{f'(x)} = \dfrac{1}{x} + x$ or equivalently $\dfrac{f'(x)}{f(x)} = \dfrac{x}{1+x^2}$.

We integrate again and get

$$\ln f(x) - \ln f(1) = \int_1^x \frac{f'(t)}{f(t)}dt = \int_1^x \frac{t}{1+t^2}dt = \frac{1}{2}\ln(1+x^2) - = \frac{1}{2}\ln 2.$$

Therefore $f(x) = \sqrt{1+x^2}$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $f : (0, +\infty) \to (0, +\infty)$ be a differentiable function that satisfies the hypothesis of the problem and let $g : (0, +\infty) \to (0, +\infty)$ be the differentiable function defined by $g(x) = \dfrac{1}{x}$. Since f is differentiable, and by the hypothesis $f'(x) = \dfrac{1}{(f\circ g)(x)}, \forall x > 0$, we conclude that f' is also differentiable and, differentiating both side of the equality $f'(x)f\left(\dfrac{1}{x}\right) = 1$, we obtain that $f''(x)f\left(\dfrac{1}{x}\right) + f'(x)f'\left(\dfrac{1}{x}\right)\dfrac{-1}{x^2} = 0$, and since $f\left(\dfrac{1}{x}\right) = \dfrac{1}{x^2}$, or equivalently, $\dfrac{(f'(x))^2 - f''(x)f(x)}{(f'(x))^2} = 1 - \dfrac{1}{x^2}$, or what is the same, $\left(\dfrac{f}{f'}\right)'(x) = 1 - \dfrac{1}{x^2}$, $\forall x > 0$.

Integrating both sides, we conclude that $\dfrac{f(x)}{f'(x)} = x + \dfrac{1}{x} + C$, $\forall x > 0$, for some $C \in \Re$. If we take $x = 1$ at the start of the inequality, and since $f(1) = \sqrt{2}$, we obtain that $f'(1) = \dfrac{1}{\sqrt{2}}$ and $\dfrac{f(1)}{f'(1)} = 2 + C$, from where $C = 0$, which implies, because $f(x) > 0$ $\forall x > 0$ by hypothesis and $\dfrac{f(x)}{f'(x)} = x + \dfrac{1}{x} + 0$ and $\dfrac{f'(x)}{f(x)} = \dfrac{x}{x^2+1}$, $\forall x > 0$.
Integrating both sides of this last equality, we conclude that $\ln(f(x)) = \log\left(\sqrt{x^2+1}\right) + D$, $\forall x > 0$ for some $D \in \Re$. Taking $x = 1$ in this equality and using the fact that $f(1) = \sqrt{2}$, we find that $D = 0$ and therefore $f(x) = \sqrt{x^2+1}$, $\forall x > 0$.

Since the function $f : (0, +\infty) \to (0, +\infty)$ defined by $f(x) = \sqrt{x^2+1}, \forall x > 0$, is differentiable with $f'(x) = \dfrac{x}{\sqrt{x^2+1}}$ and satisfies that $f(1) = \sqrt{2}$, and that

$$f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}}{\sqrt{\frac{1}{x^2}+1}} = \frac{1}{f(x)},$$

$\forall x > 0$, we conclude that the only differentiable function that satisfies the conditions of the problem is the function $f(x) = \sqrt{x^2+1}$, $\forall x > 0$.

Solution 4 by Toshihiro Shimizu, Kawasaki, Japan

We have $f'\left(\frac{1}{x}\right)f(x) = 1$. Letting x to $\frac{1}{x}$ we also have $f'(x)f\left(\frac{1}{x}\right) = 1$ $(*)$. Thus,

$$\begin{aligned}\frac{d}{dx}\left(f(x)f\left(\frac{1}{x}\right)\right) &= f'(x)f\left(\frac{1}{x}\right) + \left(-x^{-2}\right)f(x)f'\left(\frac{1}{x}\right)\\ &= 1 - x^{-2}.\end{aligned}$$

Integrating it, we have

$$f(x) f\left(\frac{1}{x}\right) = x + \frac{1}{x} + C$$

Letting $x = 1$, we have $2 = 2 + C$ or $C = 0$. Therefore $f(x) f\left(\frac{1}{x}\right) = x + \frac{1}{x}$. Multiplying $f(x)$ to $(*)$, we have

$$\begin{aligned} \left(x + \frac{1}{x}\right) f'(x) &= f(x) \\ \frac{f'(x)}{f(x)} &= \frac{1}{x + \frac{1}{x}} \end{aligned}$$

Integrating again, we have

$$\begin{aligned} \log f(x) &= \int \frac{dx}{x + \frac{1}{x}} \\ &= \int \frac{x}{x^2 + 1} dx \\ &= \frac{1}{2} \int \frac{\left(x^2 + 1\right)'}{x^2 + 1} dx \\ &= \frac{1}{2} \log\left(x^2 + 1\right) + D \end{aligned}$$

Thus, we can write $f(x) = D\sqrt{x^2 + 1}$ where D is some constant. Letting $x = 1$, we have $D = 1$. Therefore, we have $f(x) = \sqrt{x^2 + 1}$, this function actually satisfies the condition.

Also solved by Abdallah El Farsi, Bechar, Algeria; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel; Moti Levy, Rehovot, Israel; Ravi Prakash, New Delhi, India, and the proposers.